LECTURE: 5-5 THE SUBSTITUTION RULE (PART 1)

Example 1: How would we factor $x^4 - 5x^2 + 6$ and how might it relate to finding $\int 2x\sqrt{1+x^2}dx$?

$u = x^2$ so $x^4 - 5x^2 + 6$
$= u^2 - 5u + 6$
$= (u-2)(u-3)$
$= (x^2-2)(x^2-3)$

$u = 1 + x^2$
$\frac{du}{dx} = 2x$
$du = 2x\,dx$

$= \int \sqrt{u}\,du$
$= \int u^{1/2}\,du$
$= \frac{2}{3}u^{3/2} + C$
$= \boxed{\frac{2}{3}(1+x^2)^{3/2} + C}$

The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Note, the substitution rule is basically undoing the chain rule.

$\int f(g(x))g'(x)\,dx = \int f(u)\,du$

$u = g(x)$
$\frac{du}{dx} = g'(x) \Rightarrow du = g'(x)dx$

$\rightsquigarrow g(x)$ is your "inside" function.

Example 2: Evaluate $\int x^3 \cos(x^4 + 2)dx$ two different ways:

(a) solve for dx.

$u = x^4 + 2$
$du = 4x^3 dx$
$\frac{du}{4x^3} = dx$

$= \int x^3 \cos(u) \cdot \frac{du}{4x^3}$
$= \frac{1}{4}\int \cos u \; du$
$= \frac{1}{4}\sin(u) + C$
$= \boxed{\frac{1}{4}\sin(x^4+2) + C}$

(b) solve for $x^3 dx$.

$u = x^4 + 2$
$du = 4x^3 dx$
$\frac{1}{4}du = x^3 dx$

$= \int \frac{1}{4}\cos(u)\,du$
$= \frac{1}{4}\sin(u) + C$
$= \boxed{\frac{1}{4}\sin(x^4+2) + C}$

The trickiest thing about subsitution is deciding what to substitute. As substiution is (usually) undoing the chain rule you chould let your u be the inside function. Choose u to be the stuff inside of a power, root sign, denominator, or trigonometric function. When you are choosing your u the derivative of u should appear elsewhere in the integrand up to a constant multiple. The only way to get better is a lot of practice!
Once you make your subsitution the integral usually simplifies considerably. If your original variable does not completely disappear when making the subsitution you either (a) chose a subsitution that doesn't work or (b) made a mistake. At this stage you can try something different, or start your original substitution again.

Example 3: Evaluate the following indefinite integrals.

(a) $\int \sqrt{3x+2}dx = \frac{1}{3}\int u^{1/2}\,du$

$u = 3x+2$
$du = 3dx$
$\frac{1}{3}du = dx$

$= \frac{1}{3}\frac{2}{3}u^{3/2} + c$

$= \boxed{\frac{2}{9}(3x+2)^{3/2} + c}$

(b) $\int \cos^4 x \sin x dx = \int (\cos x)^4 \sin x\, dx$

$u = \cos x$
$du = -\sin x dx$
$-du = \sin x dx$

$= -\int u^4\, du$

$= -\frac{1}{5}u^5 + c$

$= \boxed{-\frac{1}{5}\cos^5 x + C}$

Example 4: Evaluate the following indefinite integrals.

(a) $\int \frac{\sec^2 x}{\tan^2 x}dx = \int \frac{1}{u^2}\,du$

$u = \tan x$
$du = \sec^2 x\, dx$

$= \int u^{-2}\,du$

$= -1u^{-1} + c$

$= \boxed{\frac{-1}{\tan x} + c}$

(b) $\int \frac{x}{\sqrt{1-x^4}}dx = \int \frac{x}{\sqrt{1-(x^2)^2}}dx$

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du$

$= \frac{1}{2}\sin^{-1} u + c$

$= \boxed{\frac{1}{2}\sin^{-1}(x^2) + c}$

Example 5: Evaluate the following indefinite integrals.

(a) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^u\,du$

$u = \sqrt{x} = x^{1/2}$
$du = \frac{1}{2}x^{-1/2}dx$
$2du = \frac{1}{\sqrt{x}}dx$

$= 2e^u + c$

$= \boxed{2e^{\sqrt{x}} + C}$

(b) $\int \frac{\arctan x}{x^2+1}dx = \int u\,du$

$u = \arctan x$
$du = \frac{1}{x^2+1}dx$

$= \frac{1}{2}u^2 + c$

$= \boxed{\frac{1}{2}(\arctan x)^2 + c}$